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The stability of steep gravity waves. Part 2

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In the previous work (Tanaka 1983), the linear stability problem of surface gravity waves on deep water to 'superharmonic' disturbances was investigated. The result obtained there suggested that the waves lose stability at the steepness which corresponds to the maximum total energy and the impulse. This result, however, apparently contradicts other work and thus the validity of it has been regarded as questionable. In the present paper, the validity of our previous result is first confirmed by two independent methods. Then, it is also shown that the contradictions with other works will disappear in a natural way when the explicit form of the unstable disturbance around the critical steepness is appropriately taken into account.

1. Introduction

In the previous work (Tanaka 1983, hereinafter referred to as [MT]) we investigated the linear stability of periodic surface gravity waves on deep water to 'superharmonic' disturbances. This problem was first treated intensively by Longuet-Higgins (1978), and by extrapolating the numerical results obtained there, he proposed that the periodic gravity waves on deep water would lose stability at the steepness ak where the phase speed of the wave attains a maximum. (As shown by Longuet-Higgins & Fox (1978), many important quantities associated to the gravity waves such as the phase speed, the total energy, the impulse do not change monotonically but oscillate infinitely many times as ak is increased toward the limiting value $ak = 0.4434$.) To check the validity of this conjecture, we investigated this same problem in [MT], where the linear stability calculation was performed with high accuracy even for those waves with large steepness that were not treated by Longuet-Higgins. The most remarkable result obtained there was that, contrary to the conjecture of Longuet-Higgins, the instability first occurred at the steepness which corresponds to a maximum not of the phase speed but of the total energy and the impulse of the basic wave. (These two quantities are known to become stationary at the same values of ak, Longuet-Higgins 1975.)

This result of [MT], however, involves some apparent contradictions with other works. Chen & Saffman (1980) first investigated the problem of bifurcation of steady gravity waves on deep water. They traced the value of the Jacobian of the system of governing equations for steady gravity waves in order to detect the bifurcation points. According to their analysis, the Jacobian never changed sign at any steepness even though they followed the solution branch for regular waves of class 1 for almost the full range of ak. They concluded that there would thus probably be no 'superharmonic' bifurcation. The problem of bifurcation of gravity waves has recently been treated again by Longuet-Higgins (1985). He employed a formulation of the problem quite different from that of Chen & Saffman and gave analytically an independent confirmation of the Chen–Saffman's guess that there is no 'superharmonic' bifurcation (other than the trivial one corresponding to a pure phase shift). On the other hand, the stability calculation of [MT] shows that there is an eigenmode (the mode $n = 2$ after the designation by Longuet-Higgins) whose eigenvalue becomes zero at the steepness which corresponds to the maximum total energy, implying that the Jacobian of Chen & Saffman should vanish there. The existence of such an eigenmode which becomes neutral at some steepness would also imply that there would be a new branch of steady solution emanating from that critical point and that a 'superharmonic' bifurcation would occur.

Moreover, quite recently, Longuet-Higgins (1984) proved analytically that, apart from the trivial eigenmode which corresponds to a pure phase shift, an eigenmode with zero eigenvalue is possible only at the steepness where the phase speed is stationary. Therefore, the existence of a neutral eigenmode as mentioned above seems to violate this analytical result because this mode is not the trivial one in the sense that $n \neq 1$ and the phase speed of the wave is not stationary at all at the point of maximum total energy.

The existence of these apparent contradictions with other work raised doubts about the validity of the numerical result of [MT] and so we have published this paper to confirm independently the validity of the numerical result of [MT] and to give a satisfactory explanation for the cause of the contradictions mentioned above.

In §2, we describe our recent numerical experiments in which the linearized disturbance equations are integrated directly with respect to the time. Then, one can check the validity of [MT] simply by comparing the growth rates of the disturbances evaluated from these experiments and those anticipated from the result of [MT]. In order to get another independent verification, we have also carried out the linear stability calculation again, employing the method developed by Longuet-Higgins (1978), a method quite different from our original one. The results obtained from these calculations will be shown in the same section.

Section 3 is devoted to the second purpose of this work. It will be shown there that the linear independence of the unstable eigenvector and the eigenvector which corresponds to a pure phase shift gradually diminishes as ak approaches the critical value, and ultimately the two become linearly dependent just at the critical point. By the use of this new and unexpected fact, the apparent contradictions mentioned above are shown to disappear and all the previous work to become consistent.

In the final section, we shall first summarize the results obtained in the present work and then briefly refer to the problems which remain as yet unresolved and await further investigations.

2. Temporal evolution of linear disturbances

As in the previous work [MT], let us introduce a variable ξ which is defined by a series of conformal mappings as follows:

$$\left.\begin{aligned} \zeta &= \exp\left(-\frac{\mathrm{i}W}{c}\right); \\ \xi &= \frac{\zeta+\alpha}{1+\alpha\zeta} \quad (-1 < \alpha \leqslant 0); \end{aligned}\right\} \tag{2.1}$$

where $W(=\Phi+\mathrm{i}\Psi)$ is the complex velocity potential of the basic undisturbed wave with phase speed c. By these transformations, the one-wave cycle of the basic wave is ultimately mapped onto the unit disk of the ξ-plane with a branch cut along the negative real axis $(-1 \leqslant \operatorname{Re}\xi \leqslant \alpha)$ as shown in figures 1 and 4 of [MT]. The last transformation $(\zeta \rightarrow \xi)$ was introduced in order to stretch the region near the crest, and the rate of stretch is intensified as the parameter α approaches -1.

Throughout this work, we consider small disturbances superposed on a steady gravity wave. Thus we shall express the displacement of the free surface $\eta(x,t)$ and the velocity potential $\phi(x,y,t)$ as

$$\eta(x,t) = H(x)+\tilde{\eta}(x,t),$$
$$\phi(x,y,t) = \Phi(x,y)+\tilde{\phi}(x,y,t),$$

where $H(x)$ and $\Phi(x,y)$ represent the displacement of the free surface and the velocity potential of the steady undisturbed wave, respectively, and $\tilde{\eta}(x,t)$ and $\tilde{\phi}(x,y,t)$ are small disturbances to these quantities. By linearizing the free-surface boundary conditions with respect to small disturbances, we obtain the following linearized disturbance equations which must be satisfied along the undisturbed free surface $y=H(x)$,

$$\left.\begin{aligned} &\tilde{\eta}_t+\Phi_x\tilde{\eta}_x+(H_x\partial_x-\partial_y)\tilde{\phi}+\tilde{\eta}(H_x\partial_x-\partial_y)\Phi_y=0,\\ &\tilde{\phi}_t+(\Phi_x\partial_x+\Phi_y\partial_y)\tilde{\phi}+\tilde{\eta}\{(\Phi_x\partial_x+\Phi_y\partial_y)\Phi_y+1\}=0.\end{aligned}\right\} \qquad (2.2)$$

Since the undisturbed free surface corresponds to the unit circle of the ξ-plane, the conditions (2.2) can be transformed into those on this unit circle as follows:

$$\left.\begin{aligned} \tilde{\eta}_t &= cq^2F(\gamma)\,\tilde{\eta}_\gamma-\tilde{u}\tan\vartheta+\tilde{v}+\frac{qF(\gamma)}{\cos\vartheta}\frac{\mathrm{d}}{\mathrm{d}\gamma}(cq\cos\vartheta)\,\tilde{\eta};\\ \tilde{\phi}_t &= -cq(\tilde{u}\cos\vartheta+\tilde{v}\sin\vartheta)+\left\{cq^2F(\gamma)\frac{\mathrm{d}}{\mathrm{d}\gamma}(cq\sin\vartheta)-1\right\}\tilde{\eta};\end{aligned}\right\} \qquad (2.3)$$

where γ is the arclength along the unit circle and

$$\tilde{u}\equiv\tilde{\phi}_x,\quad \tilde{v}\equiv\tilde{\phi}_y \quad\text{and}\quad F(\gamma)\equiv(1+\alpha^2-2\alpha\cos\gamma)/(1-\alpha^2).$$

The quantities q and ϑ describe the basic flow field and are defined by the relation $\mathrm{d}W/\mathrm{d}z = cq\,\mathrm{e}^{-\mathrm{i}\vartheta}$.

The assumption of the irrotational motion implies that the increment in $\tilde{\phi}$ in one wavelength does not depend on y and is equal to zero. (The disturbance velocity is assumed to vanish at infinite depth.) This fact means that, even though the mapped region contains a branch cut along the negative real axis, the complex velocity potential of the disturbance $\tilde{w}(\equiv\tilde{\phi}+\mathrm{i}\tilde{\psi})$ should be continuous even on the branch cut and therefore analytic throughout the unit disk without any cut. From this, we can approximate $\tilde{w}$ by a truncated MacLaurin expansion in ξ as

$$\tilde{w}\equiv\tilde{\phi}+\mathrm{i}\tilde{\psi}=\sum_{k=0}^{N}(a_k-\mathrm{i}b_k)\,\xi^k.$$

(In all the actual calculations, we set $N=128$.) Particularly along the unit circle $(\xi=\mathrm{e}^{\mathrm{i}\gamma})$, this expansion gives the following expressions for $\tilde{\phi}(\gamma)$, $\tilde{u}(\gamma)$ and $\tilde{v}(\gamma)$ along the free surface:

$$\left.\begin{aligned}
\tilde{\phi}(\gamma) &= a_0 + \sum_{k=1}^{N} (a_k \cos k\gamma + b_k \sin k\gamma);\\
\tilde{u}(\gamma) &= qF\{-\sin\vartheta \sum_{k=1}^{N} k(a_k \cos k\gamma + b_k \sin k\gamma) + \cos\vartheta \sum_{k=1}^{N} k(-b_k \cos k\gamma + a_k \sin k\gamma)\};\\
\tilde{v}(\gamma) &= qF\{\sin\vartheta \sum_{k=1}^{N} k(-b_k \cos k\gamma + a_k \sin k\gamma) + \cos\vartheta \sum_{k=1}^{N} k(a_k \cos k\gamma + b_k \sin k\gamma)\}.
\end{aligned}\right\} \tag{2.4}$$

Therefore, if $\tilde{\phi}(\gamma)$ and $\tilde{\eta}(\gamma)$ are specified for $0 \leqslant \gamma \leqslant 2\pi$ at some instant, then $\mathrm{d}\tilde{\phi}/\mathrm{d}t$ and $\mathrm{d}\tilde{\eta}/\mathrm{d}t$ can be evaluated by (2.3) and (2.4). In this manner, we can trace the temporal evolution of the disturbance. For the time stepping we used a library program called ODAM in FACOM SSL II which integrates a system of first-order ordinary differential equations by the Adams–Bashforth–Moulton scheme.

As in [MT], we employed the quantity $\omega(\equiv 1 - q_{\text{crest}}/q_{\text{trough}})$ in order to identify the basic wave. The numerical experiments were performed for five different values of ω (viz. $\omega = 0.82$, 0.83, 0.84, 0.85 and 0.86) which all lie between ω_E and ω_c, where ω_E and ω_c correspond to the total energy maximum ($\omega_E = 0.8135$; $ak = 0.4292$) and the phase speed maximum ($\omega_c = 0.8637$; $ak = 0.4359$), respectively. The result of [MT] suggests that the basic wave is unstable for all these values of ω and that the disturbance would grow exponentially in all cases. For the initial conditions for $\tilde{\phi}$ and $\tilde{\eta}$, we employed the unstable eigenfunctions obtained in the linear stability calculations of [MT]. To determine the growth rate of the disturbance, we calculated the ratio $R(t)$ of the potential energy at the initial and the later time, i.e.

$$R(t) \equiv \frac{\displaystyle\int_{-\pi}^{\pi} \{\tilde{\eta}(x,t)\}^2\,\mathrm{d}x}{\displaystyle\int_{-\pi}^{\pi} \{\tilde{\eta}(x,0)\}^2\,\mathrm{d}x}.$$

If the disturbance grows as $\mathrm{e}^{\lambda t}$, as assumed in the linear stability calculation, the growth rate λ can be calculated by $\lambda = \ln R(t)/2t$.

Table 1 lists the quantitative results for the cases of $\omega = 0.83$ and 0.85. This table clearly shows that the disturbance actually grows exponentially in time with just the growth rate anticipated from [MT]. In all the other cases, for which we do not show the results explicitly, we could also obtain such agreement.

In order to obtain another independent verification, we also carried out the linear stability calculations again, employing basically the method developed by Longuet-Higgins (1978), a method quite different from our original one.† Because the method is almost the same as his except for some minor alteration mentioned below, only the numerical results will be shown here. By the method of Longuet-Higgins, the linear stability calculation is ultimately reduced to an eigenvalue problem, $\lambda \boldsymbol{A}\boldsymbol{x} = \boldsymbol{B}\boldsymbol{x}$, where $\boldsymbol{A}$ and $\boldsymbol{B}$ are some known matrices and the eigenvalues λ correspond to the growth rate of the linear disturbances (i.e. the disturbances are assumed to depend on t as $e^{\lambda t}$). The dimension of the matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ is $4N+2$ with N the order of the truncation of the Fourier series appearing in the formulation. However, as in our original formulation, it can easily be shown that this eigenvalue problem for λ with dimension $4N+2$ can be reduced to that for λ^2 with dimension $2N+1$ by an

† The same calculation has recently also been performed independently by Professor Longuet-Higgins. He kindly gave us the results of his calculation and we could check the validity of our results.

	$\omega = 0.83$ ($\lambda_{theor.} = 1.4235 \times 10^{-1}$)		$\omega = 0.85$ ($\lambda_{theor.} = 2.2264 \times 10^{-1}$)	
t	$R(t)$	$\ln R(t)/2t$	$R(t)$	$\ln R(t)/2t$
1.0	1.3294	1.4235×10^{-1}	1.5609	2.2264×10^{-1}
2.0	1.7672	1.4235×10^{-1}	2.4365	2.2264×10^{-1}
3.0	2.3493	1.4235×10^{-1}	3.8031	2.2264×10^{-1}
4.0	3.1230	1.4235×10^{-1}	5.9365	2.2264×10^{-1}
5.0	4.1517	1.4235×10^{-1}	9.2659	2.2263×10^{-1}

TABLE 1. The quantitative results of the numerical experiments for $\omega = 0.83$ and 0.85. $\lambda_{theor.}$ are the growth rates anticipated from the result of [MT].

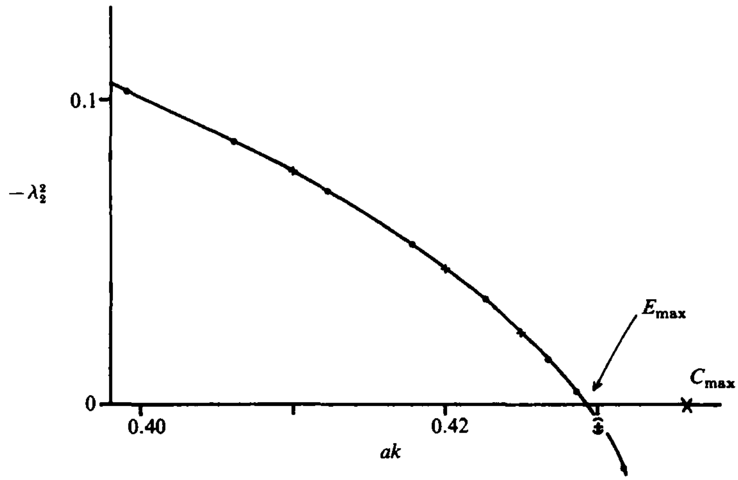

FIGURE 1. Graph of the squared eigenvalue $-\lambda^2$ versus ak. ●●●, our previous results (1983); +, present results. The bracket attached at $ak = 0.43$ roughly indicates the numerical inaccuracy involved.

appropriate rearrangement of the rows and columns. Therefore the dimension of the matrices which should be handled in the actual calculation can be reduced by half without any loss of information. Needless to say, this reduction is possible at the stage of the original linearized disturbance equations (2.2) or (2.3) before the introduction of the Fourier series expansions and the truncations.

Figure 1 shows the behaviour of the eigenvalue of the mode $n = 2$ thus obtained. The result of our previous work [MT] is also shown in the same figure for comparison. Because this method does not involve any artificial stretching of the coordinates like that involved in our original method, the accuracy of the basic wave and the convergence of the relevant eigenvalues are not sufficient especially at the largest steepness treated. Nevertheless, figure 1 still indicates clearly that the formulation of Longuet-Higgins and our original one are equivalent and give the same result.

3. The eigenvectors near the critical point

Quite recently, Longuet-Higgins (1984) has produced another great advance in the linear stability theory of surface gravity waves. In that work, he restricted his attention exclusively to those eigenmodes with zero eigenvalue and derived the conclusion that, other than a pure phase shift, a neutral eigenmode can only exist at the steepnesses where the phase speed of the wave becomes stationary. His approach was entirely analytical, involving rather simple matrix calculus only, and there seems no doubt about the validity of this conclusion.

On the other hand, the linear stability calculation of [MT] clearly indicates that, besides the mode $n = 1$ which is the trivial neutral eigenmode corresponding to a pure phase shift, there exists another neutral mode $n = 2$ at the point of maximum total energy. As the phase speed does not become stationary at this point, the existence of this extra neutral mode would seem to violate the analytical result of Longuet-Higgins mentioned above.

However, there remains one way in which the analytical result of Longuet-Higgins and the numerical result of [MT] do not contradict each other. If the eigenvector of the mode $n = 2$ happens to become identical with that of a pure phase shift (i.e. the mode $n = 1$) at the critical point, then the two results are not in conflict at all.† This unexpected possibility has not been taken into account before, but actually occurs at the critical point as shown below.

The eigenvectors $\boldsymbol{x}$ considered in this section consist of $4N+1$ elements as

$$^t\boldsymbol{x} = (a_1, \ldots, a_N;\ d_0, \ldots, d_N;\ b_1, \ldots, b_N;\ e_1, \ldots, e_N)$$

where the elements are related to $\tilde{u}$ and $\tilde{\eta}$ by the following relations,

$$\tilde{u} = \sum_{k=1}^{N} (a_k \cos k\gamma + b_k \sin k\gamma) + a_0,$$

$$\tilde{\eta} = \sum_{k=1}^{N} (d_k \cos k\gamma + e_k \sin k\gamma) + d_0.$$

The eigenvector of the mode $n = 1$ (a pure phase shift) and that of the mode $n = 2$ will be expressed as $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$, respectively. The upper half of $\boldsymbol{x}_1$ is of course zero because of the antisymmetry of this mode. All the eigenvectors are assumed to be normalized as $|\boldsymbol{x}|_2 = 1$.

As a measure of the linear independence of $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$, we employed the quantity G_{12} defined by

$$G_{12} \equiv 1 - (\boldsymbol{x}_1, \boldsymbol{x}_2)^2,$$

which is the so-called Gram's determinant for two vectors $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ and is well known to vanish only when $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ are linearly dependent. In figure 2 the values of G_{12} are shown as a function of ω. As is clearly seen from this figure, the linear independence of the two vectors diminishes as ω approaches the critical value and ultimately G_{12} vanishes just when the critical point is reached, implying that the unstable eigenvector $\boldsymbol{x}_2$ becomes identical with that of a pure phase shift $\boldsymbol{x}_1$ there. Thus the numerical result of [MT] does not violate the analytical conclusion of Longuet-Higgins.

This situation may be summarized in more general terms as follows. At the critical point (i.e. the point of maximum total energy), the algebraic multiplicity of zero eigenvalue of the relevant coefficient matrix is 4 (as the non-reduced system) while the geometrical multiplicity of that eigenvalue (i.e. the number of linearly independent eigenvectors) is only 1. The Jordan block corresponding to zero eigenvalue would therefore be of 4 by 4 with three 1's just above the main diagonal.

By taking into account this new finding, the cause of the apparent contradiction with Chen & Saffman can also be explained. Their method of calculation was by an

† This possibility was first pointed out to the author by Dr M. Yamada of Kyoto University.

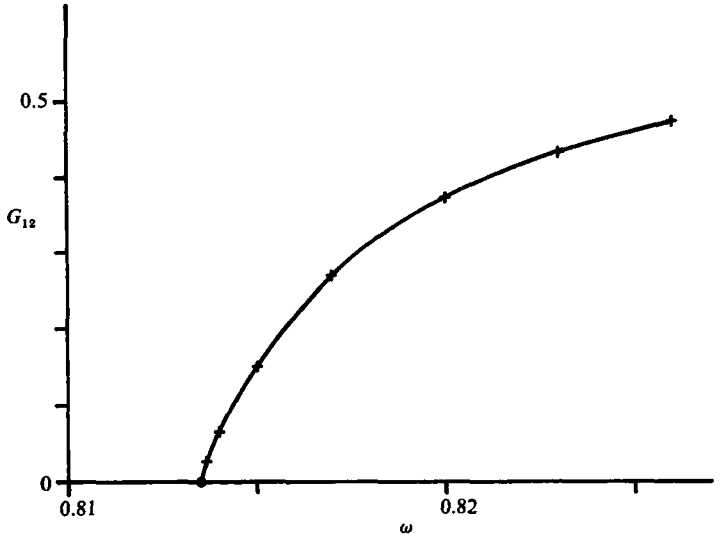

FIGURE 2. Graph of G_{12} versus ω. The sign ($\bullet$) indicates the critical point detected in [MT].

integro-differential equation involving the displacement of the free surface. Any solution of this equation, however, infinitely degenerated, that is, this equation involves infinitely many solutions all of which express just the same wave but merely displaced horizontally and vertically. Then they removed this degeneracy by adding three more requirements and fixing the origin of the coordinates by them. This implies that they discarded the mode corresponding to a pure phase shift from the outset. At the critical point, however, the eigenmode which becomes critical there is nothing but a pure phase shift as shown above and this is just the type of disturbance they removed from their formulation. Therefore, even if the Jacobian of Chen & Saffman does not vanish at the critical steepness detected in [MT], it by no means implies that these two works are inconsistent with each other.

As mentioned in the introduction, it has recently been proved analytically by Longuet-Higgins (1985) that, apart from the trivial bifurcation which corresponds to a pure phase shift and is possible at any steepness, no 'superharmonic' bifurcation occurs throughout the complete range of ak. We should therefore admit that the conclusion in [MT] as to the possibility of 'superharmonic' bifurcation must be erroneous. And we can now readily understand the reason for our arriving at this. When we found numerically in [MT] that the eigenvalue of the mode $n = 2$ becomes zero at the critical point, we naïvely imagined that the corresponding eigenvector should be different from that of the trivial mode $n = 1$ and that there would be another branch of steady solution along the direction of this vector. However, the situation which actually occurs at the critical point is somewhat more complicated than we imagined, as shown above, that is, the number of the independent eigenvectors corresponding to zero eigenvalue does not increase but remains one (a pure phase shift) even though the multiplicity is increased algebraically. In [MT], we carelessly overlooked this possibility and accordingly came to the erroneous conclusion.

4. Conclusions and discussions

In this work, we first performed two more numerical calculations which independently confirmed the numerical results of our previous work [MT] on the linear stability of periodic surface gravity waves on deep water to 'superharmonic' disturbances. Next, we showed explicitly that at the critical point the eigenvector of the mode $n = 2$ which loses stability there becomes linearly dependent on that of the mode $n = 1$, i.e. the trivial neutral mode corresponding to a pure phase shift, by evaluating the Gram's determinant for these two vectors. Taking into account this new finding, we also clarified the causes of the apparent contradictions which had existed between the results of [MT] and other works.

Our knowledge of the linear stability and the bifurcation of the steady solutions of surface gravity waves has thus been greatly increased by recent excellent work by several authors as well as ours. However, there still remain several problems as yet unresolved, the following two being of particular importance.†

First, although the simple intuitive argument (Longuet-Higgins 1978) and the analytical approach (Longuet-Higgins 1984) both suggest that there should exist a non-trivial neutral eigenmode at the point of maximum phase speed, nevertheless such a mode does not appear in the linear stability calculation either by the method of Longuet-Higgins or by our original one. The reason for this discrepancy is not known to the author as yet.

The second concerns the relation between the critical steepness and the steepness where the total energy and the impulse become stationary. As far as the numerical results are concerned, these two points coincide with each other and are indistinguishable. Recently we have also confirmed that the next higher mode $n = 3$ becomes unstable just at the second extremum (local minimum) of the total energy and the impulse. However, any result obtained numerically contains an inevitable lack of certainty, and some analytical proof should be necessary. The necessity for an analytical approach would be more essential in the case of the solitary wave.

We are now investigating the linear stability of the solitary wave also. Although we have not performed calculations intensively as yet and the numerical data are not sufficient, the results obtained so far clearly indicate that, like the periodic wave on deep water treated here, the solitary wave also loses stability at an amplitude which is well below the one corresponding to the maximum phase speed. In this case, however, the behaviour of the eigenvalue is quite different from that of the periodic-wave case and the numerical results appear to indicate that it would be much more difficult to decide the critical amplitude by numerical calculations only and that some analytical approach would be indispensable in doing so.

The author wishes to express his cordial thanks to Prof. M. S. Longuet-Higgins for many helpful discussions. In particular the numerical calculation reported in the latter part of §2 was performed in correspondence with him. He is also grateful to Dr M. Yamada of Kyoto University for his many valuable advices, helpful discussions and kind inspection of the manuscript. This work was supported by the Grant-in-Aid for Scientific Research 59740172 from the Ministry of Education, Science and Culture.

† Quite recently, these problems have been completely solved analytically by Saffman (1985) in quite an elegant manner.

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